FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.
- Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show behaviour at limits.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. It seemed that all were able to complete the paper in the time allowed. Sometimes candidates did not fully justify their answers and jumped to conclusions without justification, particularly where answers were given within the question. There were many scripts of an extremely high standard.

Comments on specific questions

Question 1

This was very well done with most candidates accurately evaluating the coordinates, though a few found $\int_0^1 xy\,dx$ for $\bar{y}$.

Question 2

Most candidates showed good knowledge of the structure of an induction proof, though some did not communicate all the steps clearly. Sometimes the proposition was assumed for *every* integer and a few made errors when differentiating the *k*th derivative. Most manipulated the expressions well and good candidates made the implication explicit in the final statement.

Question 3

(i) Almost all applied integration by parts to I_n. Those who started with I_{n+2} also successfully derived the given reduction formula.

(ii) This part was well done with the majority of candidates accurately applying the reduction formula.

Question 4

(i)–(ii) Apart from a small number of errors in division, these parts of the question were well done.

(iii) Stronger candidates produced well drawn and labelled sketches, showing correct forms at infinity, and curves approaching asymptotes. Some sketches had misplaced branches, but most did have the general shape correct.

Question 5

(i) The majority of candidates successfully applied standard results to fully justify the given answer.

(ii) This part was done to a high standard with candidates writing out enough terms to justify cancelation, and most remembering to give their answer in terms of *N* not *n*.

(iii) Better responses included division of S_N by N^3 before finding the limit.

Question 6

(i) The majority of candidates found the common perpendicular and applied the formula for the shortest distance accurately. A few candidates took the longer approach of finding points of intersection with the common perpendicular which produced more errors.

(ii) It was pleasing to see that candidates knew how to use a point on the plane and the substitution was often mentioned explicitly. A more successful approach involved taking the cross product of correctly chosen vectors to find a normal to the plane.

Question 7

(i) Almost all candidates substituted for x into the original equation and squared correctly to verify the result. Better responses explained the relationship between the substitution and the new roots, giving enough detail using the product of the old roots.

(ii) Most candidates correctly used the formula for the sum of squares. Other approaches seen were less successful.

(iii) Several methods were employed in the final part, with a few candidates successfully working with the original equation in terms of x. Most candidates used the given cubic in terms of y as intended. Some candidates trying to recall complicated sigma formulae made errors.

Question 8

(i) This part of the question was well done, though a few candidates accepted zero eigenvectors without checking for errors in their working.

(ii) Most candidates showed that $\mathbf{M}^7\mathbf{P} = \mathbf{PD}^7$ and could use their previous results accurately.

Question 9

(i) The majority of candidates used de Moivre's theorem successfully to find $\sec 6\theta$ with just a few struggling to translate their expression in cos and sin into an expression in sec. There were some elegant uses of $\tan^2\theta = \sec^2\theta - 1$.

(ii) More efficient responses made the connection with the equation given in the first part and, remembering that sec is an even function, used $\sec 6\theta = 2$ to find all six solutions.

Question 10

(i) Row reductions were almost always accurate, although some candidates did not reach row echelon form before deducing the rank of $\mathbf{A}$.

(ii)–(iii) Most candidates performed row operations to both sides of the system of equations and, from $(1+\theta)z = (1+\theta)$, obtained the solution accurately. Some mistakenly tried to find a general solution for **(ii)** using parameters.

(iv) Better responses used row operations to deduce that $(1+\theta)z = (1+\phi)$ and were able to fully justify that the system is inconsistent.

Question 11 – EITHER

This was the more popular choice.

(i) Accurate solutions came from substituting $\frac{dw}{dx}$ and $\frac{d^2w}{dx^2}$ to obtain the original equation. Those who worked with $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ were more prone to making errors.

(ii) Almost all candidates knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the constants and some problems with notation. A few candidates finished with w in terms of x, and some gave expressions instead of equations as their answer.

Question 11 – OR

(i) Many of those who tackled this option had difficulty with this part, not appreciating the need to factorise $e^{2\alpha} - e^{-2\alpha}$. Most could work with the given quadratic in e^{α} to find the exact values of α and r at P.

(ii) Most candidates sketched the correct shape of C_1 with the correct domain. Better responses had r strictly increasing and the correct intersection point with C_2.

(iii) More successful responses correctly subtracted the area bounded by C_2 and $\theta = \alpha$ from the area bounded by C_1 and $\theta = \alpha$. Those who formed the wrong integral by having C_1 and C_2 interchanged in this subtraction often still gained credit for expanding and integrating either expression.

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

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FURTHER MATHEMATICS

Paper 9231/13
Paper 13

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FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

To score full marks in this paper candidates must be well versed in both Mechanics and Statistics. Any preference between these two areas can only be exercised in the choice of the final optional question.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

In Mechanics questions, a diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen. As always, more candidates opted for the Statistics option then the Mechanics option in **Question 11**.

Previous reports have stressed the need for candidates to set out their work clearly, and this advice has been heeded by most. This is particularly important in the longer, unstructured questions such as **Question 2**, **Question 3** and **Question 9**.

The rubric for this paper specifies that non-exact numerical answers be given to 3 significant figures. Candidates would therefore be well advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one.

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Question 1

Many candidates made a very good attempt at this question. The most common error was to quote the formula for the radial acceleration correctly as $\frac{v^2}{r}$ but then substitute for v instead of v^2. A minority of candidates confused the formulae for radial and transverse accelerations.

Question 2

As in all questions of this type, candidates are well advised to first identify all the forces acting on the lamina, preferably showing them on the given diagram. This will help the candidate to ensure that they include all the relevant forces when taking moments or resolving. Candidates were required to find the normal reaction forces at *E* and *B*. No method was suggested, so the candidate was free to choose the approach. The most direct method of solution is to take moments about the point *B*, thereby eliminating all the forces except the reaction at *E* and the weight acting at the centre of the lamina. The second of these proved troublesome to many candidates, who had difficulty in finding the distance of the line of action of the weight from the point *B*.

The simplest way was to resolve the weight into two components, one parallel to *AD* and one perpendicular to *AD*. Alternatively, a distance involving a sine or cosine of a sum of two angles, and its expansion, is required. Using this approach, the next step is to resolve the forces vertically, leading to an expression for the normal reaction force at *E,* and then horizontally to find the frictional force.

Of course it is possible to take moments about several other points, and many candidates did indeed do so. Almost invariably, however, the candidates who did this were unsuccessful in isolating the forces that they required from the resulting moments equations.

Question 3

Almost all candidates were able to formulate two simultaneous equations for the speeds of *A* and *B* after the first collision, by means of conservation of momentum and Newton's law of restitution. There were a few sign errors in the equations. Candidates are reminded that a diagram with masses and velocities, with magnitude and direction clearly marked, is invaluable in avoiding such errors. Having found these speeds, the process then needs to be repeated for the second collision, between *B* and *C*.

Many candidates made the second part of the question much more difficult than it needed to be. After the collision between *B* and *C*, those spheres cannot collide again, so to guarantee no further collisions between any of the spheres, the only necessary condition is that the speed of *B* after the second collision is greater than the speed of *A* after the first collision. This leads to a quadratic inequality involving the coefficient of restitution: $\frac{1}{3} \leqslant e \leqslant 3$. The final step is to realise that $e \leqslant 1$, so $\frac{1}{3} \leqslant e \leqslant 1$.

Question 4

Most candidates were able to formulate two equations, one using Newton's second law of motion and a second by applying conservation of energy from *A* to *B*, from the initial position to the point where the string goes slack. There were some sign errors, which led to some dubious algebra in showing the given result.

In the second part, the particle behaves as a projectile with its initial velocity equal to its speed at the moment the string goes slack. It is then necessary to consider the subsequent vertical motion. The common error was to use the speed at the moment the string goes slack, rather than its vertical component.

Question 5

Almost all candidates were able to make an attempt at finding the moment of inertia of the object and did so by finding the moment of inertia for each of the three component parts; the rod, the hollow sphere and the solid sphere. All three of these required the use of the parallel axes theorem. Many candidates set out their work clearly, with each of the contributions to the total moment of inertia clearly identified. Such detail is important when the final result is given in the question. Candidates who simply write down a sum of terms run considerable risk, since an error in one term that still leads to the given correct answer does cast doubt on the validity of their whole process.

The second part of this question posed problems for some candidates, with a minority making no real attempt. The intended method of solution required application of the result $C = I\ddot{\theta}$, to obtain an equation representing simple harmonic motion, for small oscillations of the system. Those who pursued this method were often successful, although unsupported minus signs appeared in a significant number of cases.

Question 6

Almost all candidates knew how to proceed with both parts of this question, but a significant minority did not realise that they needed to use a *t*-value, because a small sample was involved.

Question 7

This question on the negative exponential distribution produced good answers by many candidates. In the final part, the equation to be solved is $1 - p^n > 0.99$. A common error was to have q (= $1 - p$) rather than p. The method of solution of the equation, using logarithms, was usually known and well executed.

Question 8

Many candidates scored full marks for this question. Common errors were using an incorrect form of the pooled variance and/or using an incorrect tabular *t*-value. A minority of candidates did not use the information that the distributions of the weights of the elephants in the two regions should be assumed to have the same population variance. This was a stated assumption and should have been used as information by the candidate.

Question 9

The majority of candidates used the given information and known formulae to find the correct values of p and q. The algebra involved needed care, and it was pleasing to see a high degree of accuracy.

Question 10

The distribution function of X is found by integrating the given probability density function. The majority of candidates performed the integration correctly, but a significant number did not consider the endpoints of the interval, $2 \leqslant x \leqslant 4$. It is necessary to include a constant of integration to ensure that the distribution function is equal to 0 for $x \leqslant 2$ and equal to 1 for $x \geqslant 4$. The form/values of the distribution function must be specified for all values of x.

The given transformation is then applied to find the distribution function and the probability density function of Y. Again, this was usually done accurately.

Question 11 (Mechanics)

This optional question was attempted by just under one-third of candidates, and the solutions were often of a very good standard. The first result is obtained by equating the tensions in the two strings. In the second part, it is necessary to form a differential equation, from an application of Newton's second law, and from the form of this simplified differential equation to deduce that the motion is simple harmonic. The remaining parts of the question require the use of some of the expressions for the period, the velocity and the displacement for simple harmonic motion.

Question 11 (Statistics)

This question tests the appropriateness of a Poisson distribution as a fit to the given data. In the first part, the mean and standard deviation of the given data are calculated as 1.7 and 1.56 respectively. Since these are similar, it can be deduced that the Poisson distribution may be a suitable fit for the data. All but two of the expected frequencies were given in the table, and candidates were asked to verify just one of them. The second could be calculated by summing all the expected frequencies to one. Most candidates were able to do this.

A goodness of fit test is then carried out, and most candidates showed knowledge of the basic method for this. However, there are several expected frequencies in the table that are less than 5, and the last 4 entries must be summed before the chi-squared values are evaluated. A significant minority of candidates omitted to do this. The calculated chi-squared value 1.25 should be compared with the tabular value 6.251, leading to acceptance of the null hypothesis.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

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This question tests the appropriateness of a Poisson distribution as a fit to the given data. In the first part, the mean and standard deviation of the given data are calculated as 1.7 and 1.56 respectively. Since these are similar, it can be deduced that the Poisson distribution may be a suitable fit for the data. All but two of the expected frequencies were given in the table, and candidates were asked to verify just one of them. The second could be calculated by summing all the expected frequencies to one. Most candidates were able to do this.

A goodness of fit test is then carried out, and most candidates showed knowledge of the basic method for this. However, there are several expected frequencies in the table that are less than 5, and the last 4 entries must be summed before the chi-squared values are evaluated. A significant minority of candidates omitted to do this. The calculated chi-squared value 1.25 should be compared with the tabular value 6.251, leading to acceptance of the null hypothesis.

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

To score full marks in this paper candidates must be well versed in both Mechanics and Statistics. Any preference between these two areas can only be exercised in the choice of the final optional question.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

In Mechanics questions, a diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen. As always, more candidates opted for the Statistics option then the Mechanics option in **Question 11**.

Previous reports have stressed the need for candidates to set out their work clearly, and this advice has been heeded by most. This is particularly important in the longer, unstructured questions such as **Question 2**, **Question 3** and **Question 9**.

The rubric for this paper specifies that non-exact numerical answers be given to 3 significant figures. Candidates would therefore be well advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one.

Comments on specific questions

Question 1

Many candidates made a very good attempt at this question. The most common error was to quote the formula for the radial acceleration correctly as $\frac{v^2}{r}$ but then substitute for v instead of v^2. A minority of candidates confused the formulae for radial and transverse accelerations.

Question 2

As in all questions of this type, candidates are well advised to first identify all the forces acting on the lamina, preferably showing them on the given diagram. This will help the candidate to ensure that they include all the relevant forces when taking moments or resolving. Candidates were required to find the normal reaction forces at *E* and *B*. No method was suggested, so the candidate was free to choose the approach. The most direct method of solution is to take moments about the point *B*, thereby eliminating all the forces except the reaction at *E* and the weight acting at the centre of the lamina. The second of these proved troublesome to many candidates, who had difficulty in finding the distance of the line of action of the weight from the point *B*.

The simplest way was to resolve the weight into two components, one parallel to *AD* and one perpendicular to *AD*. Alternatively, a distance involving a sine or cosine of a sum of two angles, and its expansion, is required. Using this approach, the next step is to resolve the forces vertically, leading to an expression for the normal reaction force at *E,* and then horizontally to find the frictional force.

Of course it is possible to take moments about several other points, and many candidates did indeed do so. Almost invariably, however, the candidates who did this were unsuccessful in isolating the forces that they required from the resulting moments equations.

Question 3

Almost all candidates were able to formulate two simultaneous equations for the speeds of *A* and *B* after the first collision, by means of conservation of momentum and Newton's law of restitution. There were a few sign errors in the equations. Candidates are reminded that a diagram with masses and velocities, with magnitude and direction clearly marked, is invaluable in avoiding such errors. Having found these speeds, the process then needs to be repeated for the second collision, between *B* and *C*.

Many candidates made the second part of the question much more difficult than it needed to be. After the collision between *B* and *C*, those spheres cannot collide again, so to guarantee no further collisions between any of the spheres, the only necessary condition is that the speed of *B* after the second collision is greater than the speed of *A* after the first collision. This leads to a quadratic inequality involving the coefficient of restitution: $\frac{1}{3} \leqslant e \leqslant 3$. The final step is to realise that $e \leqslant 1$, so $\frac{1}{3} \leqslant e \leqslant 1$.

Question 4

Most candidates were able to formulate two equations, one using Newton's second law of motion and a second by applying conservation of energy from *A* to *B*, from the initial position to the point where the string goes slack. There were some sign errors, which led to some dubious algebra in showing the given result.

In the second part, the particle behaves as a projectile with its initial velocity equal to its speed at the moment the string goes slack. It is then necessary to consider the subsequent vertical motion. The common error was to use the speed at the moment the string goes slack, rather than its vertical component.

Question 5

Almost all candidates were able to make an attempt at finding the moment of inertia of the object and did so by finding the moment of inertia for each of the three component parts; the rod, the hollow sphere and the solid sphere. All three of these required the use of the parallel axes theorem. Many candidates set out their work clearly, with each of the contributions to the total moment of inertia clearly identified. Such detail is important when the final result is given in the question. Candidates who simply write down a sum of terms run considerable risk, since an error in one term that still leads to the given correct answer does cast doubt on the validity of their whole process.

The second part of this question posed problems for some candidates, with a minority making no real attempt. The intended method of solution required application of the result $C = I\ddot{\theta}$, to obtain an equation representing simple harmonic motion, for small oscillations of the system. Those who pursued this method were often successful, although unsupported minus signs appeared in a significant number of cases.

Question 6

Almost all candidates knew how to proceed with both parts of this question, but a significant minority did not realise that they needed to use a *t*-value, because a small sample was involved.

Question 7

This question on the negative exponential distribution produced good answers by many candidates. In the final part, the equation to be solved is $1 - p^n > 0.99$. A common error was to have q (= $1 - p$) rather than p. The method of solution of the equation, using logarithms, was usually known and well executed.

Question 8

Many candidates scored full marks for this question. Common errors were using an incorrect form of the pooled variance and/or using an incorrect tabular *t*-value. A minority of candidates did not use the information that the distributions of the weights of the elephants in the two regions should be assumed to have the same population variance. This was a stated assumption and should have been used as information by the candidate.

Question 9

The majority of candidates used the given information and known formulae to find the correct values of p and q. The algebra involved needed care, and it was pleasing to see a high degree of accuracy.

Question 10

The distribution function of X is found by integrating the given probability density function. The majority of candidates performed the integration correctly, but a significant number did not consider the endpoints of the interval, $2 \leqslant x \leqslant 4$. It is necessary to include a constant of integration to ensure that the distribution function is equal to 0 for $x \leqslant 2$ and equal to 1 for $x \geqslant 4$. The form/values of the distribution function must be specified for all values of x.

The given transformation is then applied to find the distribution function and the probability density function of Y. Again, this was usually done accurately.

Question 11 (Mechanics)

This optional question was attempted by just under one-third of candidates, and the solutions were often of a very good standard. The first result is obtained by equating the tensions in the two strings. In the second part, it is necessary to form a differential equation, from an application of Newton's second law, and from the form of this simplified differential equation to deduce that the motion is simple harmonic. The remaining parts of the question require the use of some of the expressions for the period, the velocity and the displacement for simple harmonic motion.

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A goodness of fit test is then carried out, and most candidates showed knowledge of the basic method for this. However, there are several expected frequencies in the table that are less than 5, and the last 4 entries must be summed before the chi-squared values are evaluated. A significant minority of candidates omitted to do this. The calculated chi-squared value 1.25 should be compared with the tabular value 6.251, leading to acceptance of the null hypothesis.